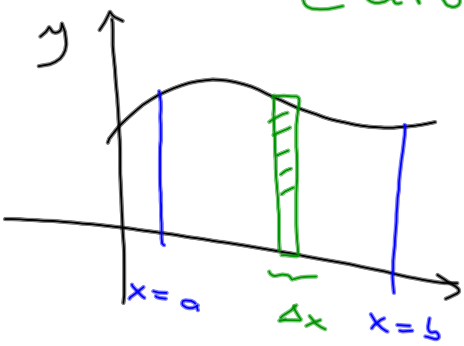
7.1 Area between Curves



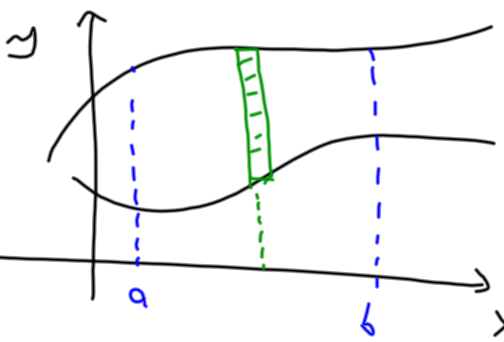
$x = g(x) = 0$

$$\int_a^b f(x)\,dx = A$$

(axis)

$$= \int_a^b \left(f(x) - \underset{=0}{g(x)}\right) dx$$

In general,

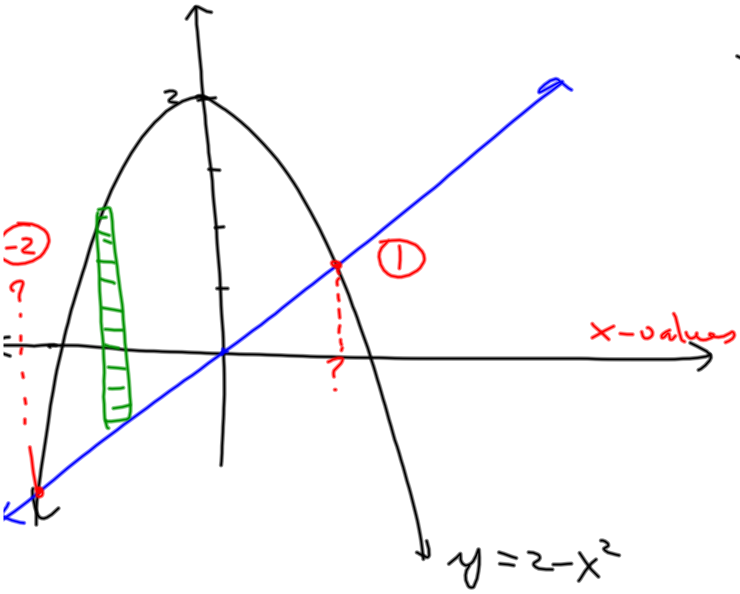


$g(x) \le f(x)$ over $[a, b]$, then the area between $g(x)$ and $f(x)$ is

$$\boxed{\int_a^b \left[f(x) - g(x)\right] dx}$$

Eg: Find the area of the region between ~~by~~ the curves: $y = 2-x^2$

$y = x$



Remarks:

1. Integrating with respect to x-values: vertical rectangular reprensentatives

2. Find the points of intersection of both curves: $y=2-x^2 = x$, solve for x. $x^2+x-2=0$

$(x+2)(x-1)=0$

$x=-2, +1$

3. Set up the integral

4. evaluate!

(3) Setup: $\int_{-2}^{1}\left[(2-x^2)-(x)\right]dx$

(4) Evaluate the above integral (when asked!)

$$A = \int_{-2}^{1}(-x^2-x+2)\,dx = \left[-\frac{1}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^{+1}$$

$$= -\frac{1}{3}-\frac{1}{2}+2-\left(+\frac{8}{3}-2-4\right) = \frac{27}{6}$$

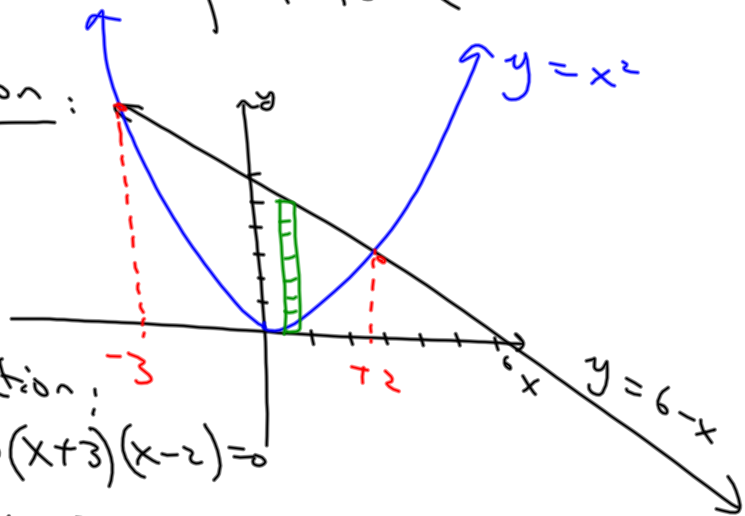
$$\boxed{A = \frac{9}{2}\ \text{unit}^2}$$

Example: Consider the following

$$y = x^2$$
$$y = 6 - x$$

(a) Find the area of the region by integrating with respect to x

Solution:



(a)

* points of intersection:

$x^2 + x - 6 = 0 \longrightarrow (x+3)(x-2) = 0$

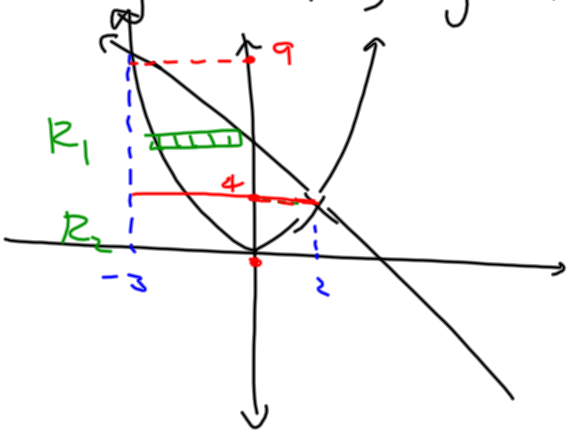
$\longrightarrow x = -3, +2$

$$A = \int_{-3}^{2} \left[6 - x - x^2\right] dx = \left[-\frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x\right]_{-3}^{2}$$

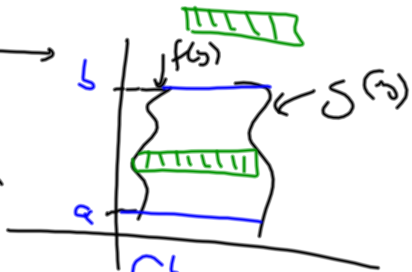
$$= \left[-\frac{8}{3} - 2 + 12 - \left(+9 - \frac{9}{2} - 18\right)\right]$$

$$= 125/6$$

⑤ Integrate the region with respect to y, while finding the area between the curves: $y = 6 - x$; $y = x^2$



horizontal representative:



$$\int_a^b \left(g(y) - f(y)\right) dy$$

values of y

Notes:

① When $x = -3$, $y = x^2 \longrightarrow y = 9$

$x = +2$, $y = 4$

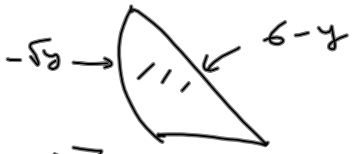
$x = 0$, $y = 0$

② Functions of y:

$y = 6 - x \longrightarrow x = 6 - y$

$y = x^2 \longrightarrow x = \pm\sqrt{y}$, $y \geq 0$

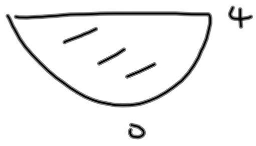
R_1:



$$\int_4^9 \left[(6-y) - (-\sqrt{y})\right] dy$$

R_2:

$$\int_0^4 \left[(\sqrt{y} - (-\sqrt{y}))\right] dy$$



$$A = \left[6y - \tfrac{1}{2}y^2 + \tfrac{2}{3}y^{3/2}\right]_4^9 + \left[2\left(\tfrac{2}{3}y^{3/2}\right)\right]_0^4 = 125/6$$

Ex: Consider the following:

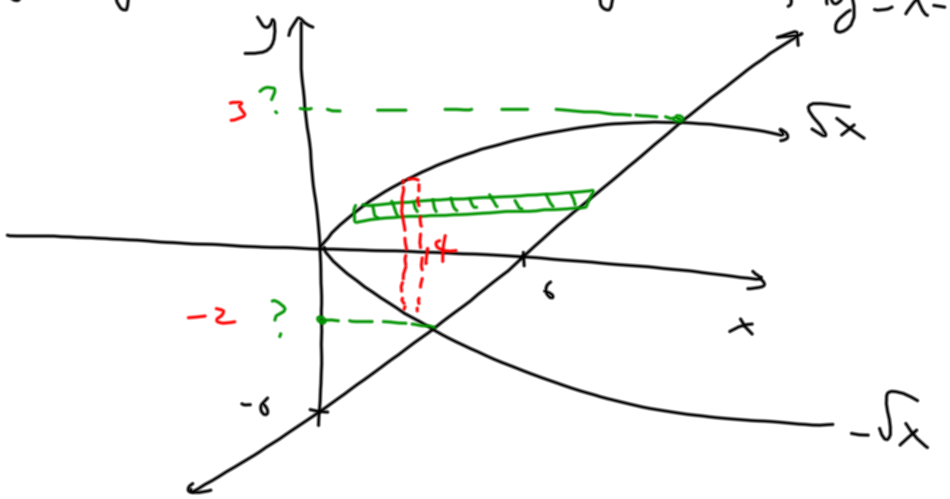
$$\begin{cases} f(y) = y^2 \\ g(y) = y + 6 \end{cases}$$

Sketch the bounded region and find the area.

Solution

$f(y) = y^2$ To sketch: Consider $x = y^2 \rightarrow y = \pm\sqrt{x}\,,\ x \geq 0$

$g(y) = y + 6$: Consider $x = y + 6 \longrightarrow y = x - 6$



points of intersection: $y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$\therefore y = 3, -2$

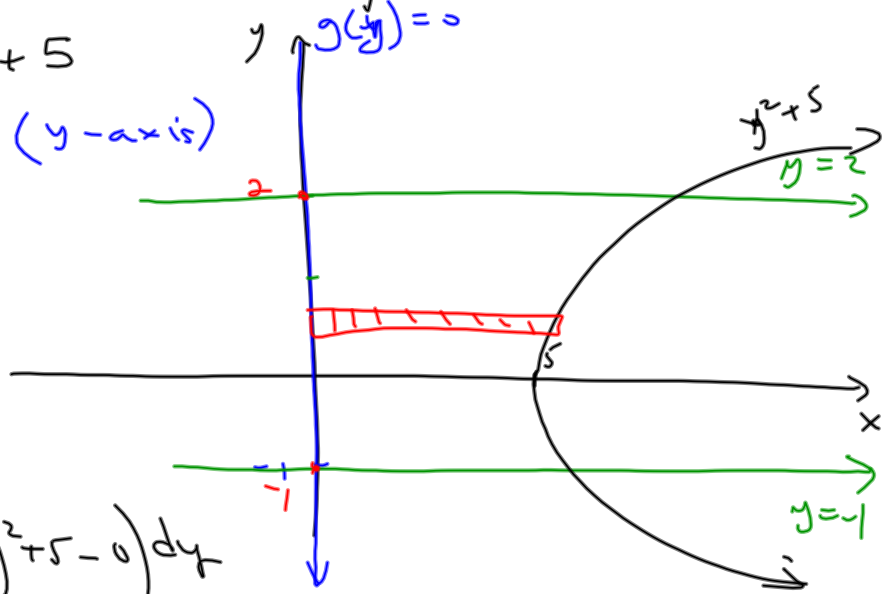
Area: $\displaystyle\int_{-2}^{3} \left(y + 6 - y^2\right) dy$

$A = 125/6$

Example:

Sketch and setup the area.

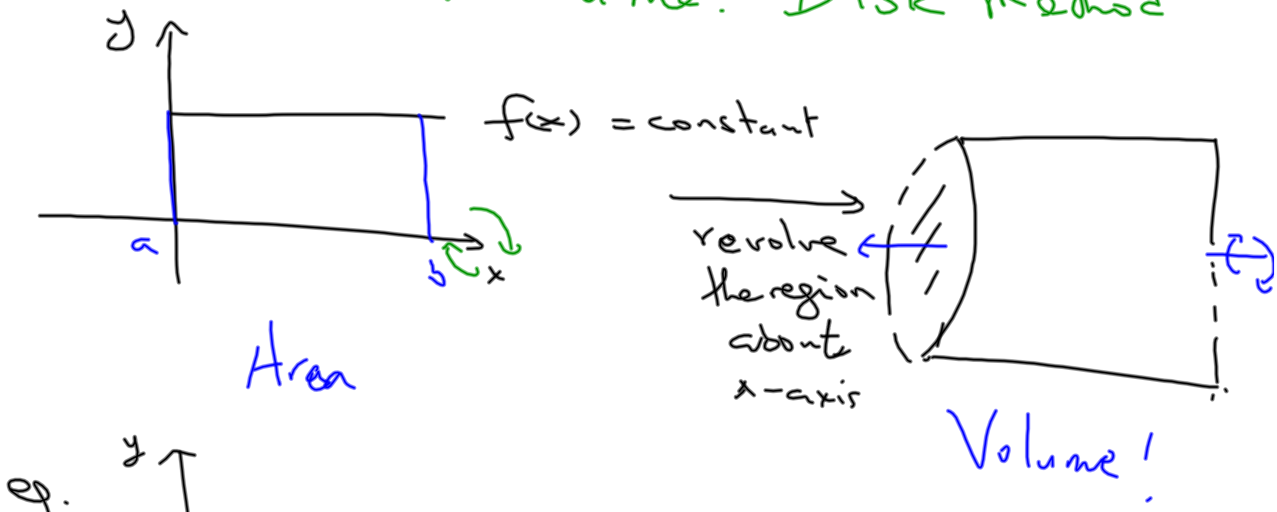
$$\begin{cases} f(y) = y^2 + 5 \\ g(y) = 0 \ (y\text{-axis}) \\ y = -1 \\ y = 2 \end{cases}$$



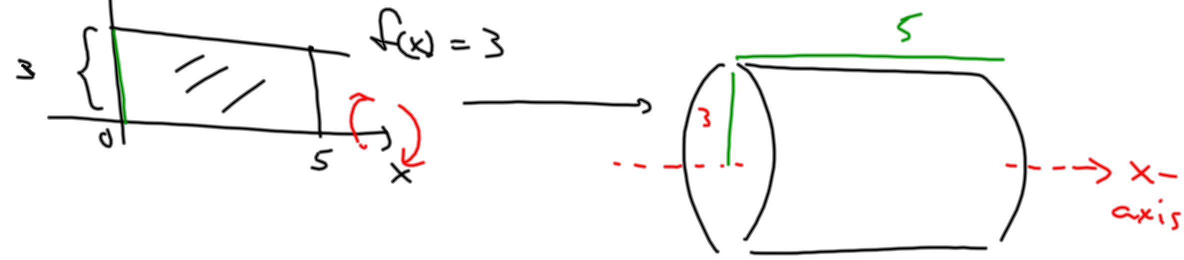
$$\text{Area} = \int_{-1}^{2} (y^2 + 5 - 0)\, dy$$

$$= \left(\frac{1}{3}y^3 + 5y\right]_{-1}^{2} = \left[\frac{8}{3} + 10 - \frac{1}{3} - 5\right] = 18 \text{ unit}^2$$

7.2 Volume: Disk method



e.g.:



Volume: $\pi r^2 h = \pi(3)^2(5)$
(Geometry) $= 45\pi$ unit^3

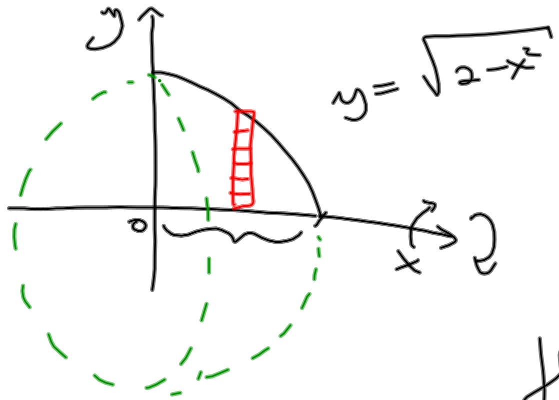
Volume: $\pi \displaystyle\int_0^5 [\text{radius}]^2\,dx$
(Calculus)

$$= \pi\int_0^5 [f(x)]^2\,dx = \pi\int_0^5 [3]^2\,dx = \pi\Big[9x\Big]_0^5$$

$$= \pi\big[9(5) - 9(0)\big]$$

$$= 45\pi \text{ unit}^3$$

Example:



Find the volume of the solid obtained by revolving the region $\begin{cases} y = \sqrt{2-x^2} \\ x = 0 \\ y = 0 \end{cases}$ about the x-axis!

$$\text{Volume} = \pi \int_0^{\sqrt{2}} \left[\sqrt{2-x^2}\right]^2 dx \qquad \text{Volume} = \pi \int_a^b \left[f(x)\right]^2 dx$$

$$= \pi \int_0^{\sqrt{2}} (2-x^2)\, dx = \pi \left[2x - \frac{1}{3}x^3\right]_0^{\sqrt{2}}$$

$$= \pi \left[2\sqrt{2} - \frac{2}{3}\sqrt{2} - 0\right]$$

$$= \pi\, 2\sqrt{2}\left(\frac{2}{3}\right) = \frac{4\pi\sqrt{2}}{3} \text{ unit}^3$$